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Constraints for Interpretation of Line Drawings under Perspective Projection*

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Problems of surface orientation recovery from line drawings in a single image obtained under perspective projection are studied. Two constraints, the shared boundary constraint and the orthogonality constraint, previously used in orthographic projection, are extended to perspective projection. New constraints are derived from observations of parallelism. We define a new symmetry, called *convergent symmetry.* Convergent symmetry results from perspective projection of a symmetric object. Unlike skew symmetry, convergent symmetry provides sufficient constraints to recover unique surface orientations. The set of techniques given should allow extension of all previous orthographic analysis and provide new tools for additional, more constrained analysis. An example illustrating the use of our techniques is provided. Finally, extension of the constraints for some class of curved surfaces is discussed. © 1991 Academic Press, Inc.

1. INTRODUCTION

Inferring the 3-D shape of an object by using only the contours of its image has been a problem of prime interest in computer vision. The problem is, of course, inherently ambiguous as many objects can give rise to the same contours under different imaging conditions. However, some constraints on the 3-D positions and orientations do follow from the image, and even unique answers can be found in some cases if certain regularity assumptions can be made. This paper provides some new constraints that apply in the case of perspective projection.

The problem of line interpretation was first studied in depth for scenes of polyhedra. Pioneering contributions were made by Huffman [1] and Clowes [2] in rules for line labeling. Mackworth [3] derived some quantitative constraints on orientations of the planes in the line drawings, based partly on work of Huffman. Kanade [4] showed how incorporating some regularity constraints, particularly symmetry constraints, allowed for unique interpretations in some cases. More recently Sugihara [5] has developed a comprehensive set of techniques to analyze polyhedral scenes.

Several researchers have also attempted to develop methods that apply to non-polygonal and non-planar curves and surfaces [6–12]. In [12] we provide a comprehensive review of these techniques and some new techniques for analysis of a broad class of curved surface scenes.

Usually, such analysis assumes *orthographic* projection. In this paper, we derive constraints that apply to the more general *perspective* projection. It has been conventionally thought that constraints for perspective projection would be too unwieldy as the image appearance depends not only on the viewing angle but also on the viewing position. We show that the resulting constraints, though more complex, are quite usable. Orthographic projection may be a good enough approximation when the viewing angle is small but perspective analysis may be necessary in other cases. However, we find that the perspective constraints actually carry *more* information and can provide more constrained or even unique interpretations.

Some researchers have investigated perspective projection before. Draper [13] gave a constraint that derives from boundary between two faces. Sugihara [5] gives a linear programming method to determine the realizability of a line drawing under orthographic projection. His formulation can also be carried out under perspective projection. However, this method leaves many degrees of freedom undetermined for surface orientations and does not provide a way of incorporating other geometric constraints like symmetries. Sugihara does show how to use additional constraints such as shape from shading in an optimization scheme.

In this paper, we provide a set of techniques for perspective projection that parallel many of the traditional techniques for orthographic projection and hence can be applied where the latter techniques apply. Some of the constraints we describe have been previously presented by Shafer, Kanade, and Kender [14]; we will make specific references to their work in the appropriate places.

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Barnard [15, 16] also studied perspective projection and used the parallelism and orthogonality of the lines in a different and interesting way.

In Section 2, we define some terms for perspective projection that are used to develop the constraints for perspective projection. Some of these constraints are generalizations of the constraints for orthographic projection, however, some are new and apply to perspective projection only. One of our contributions is the definition of a new kind of symmetry in the image that we call *convergent symmetry* that can provide unique orientations for such figures directly without using any other constraints? In Section 4, we show how to apply the mathematical constraints we have derived to an example. In Section 5, we give an analysis for curved surfaces.

2. PERSPECTIVE PROJECTION

Perspective projection is the exact projection model for the pinhole camera and a very good approximation for the lens systems used on cameras for objects in focus. In perspective projection there is a focal point; let it be the origin of the coordinate system. Any point, (x, y, z), in 3-D forms a ray passing through the point and the focal point (the origin). We can represent this ray as $(u, v, 1)$ where $u = x/z$ and $v = y/z$. The intersection of this ray with the $z = 1$ plane forms the image of this point, then the intersection has coordinates (u, v) on that plane. Note that any point (u, v) on the $u - v$ plane is also a point or a vector, $(u, v, 1)$, in the $x - y - z$ coordinate system. This duality of the points will be used throughout the paper. In the paper the capital italic letters are used to denote the vectors and points in 3-D, and the $z = 1$ plane is called the image plane, the projective plane or the $u - v$ plane.

Consider a line, $L = Rt + P$, in 3-D, parameterized in t, where $R = (r_x, r_y, r_z)$ is the orientation of the line and $P = (p_x, p_y, p_z)$ is any point on the line. From just the image of L on the projective plane we can not recover the parameters R or P. But we can extract some other useful parameters. The image, $L_i = R_i t + P_i$, of the line L can be considered as a line in 3-D which lies on the $u - v$ plane. Say L_i has equation $au + bv + c = 0$ on the $u - v$ plane; then

$$R_i = (-b, a, 0) \quad P_i = (p_u, p_v, 1), \tag{1}$$

where P_i is any point on the line L_i and R_i is the 3-D orientation of the line L_i. Note that R_i and P_i are the only observable quantities of the line L from the image of it, and they are not unique but scalable.

We define another useful observable. If the line L does not pass through the origin[1] then we can define a plane by the line L and the origin. This plane has the property that the image, L_i, of the line L is the intersection of this plane and the image plane. This plane will be called the *image generating plane* or IGP (this plane is called the interpretation plane by Macworth [3]. The IGP of L can also be constructed by using the line L_i and the origin. The normal of IGP (called the NIGP of the line L) A is given by:

$$\begin{aligned} A = R \times P &\equiv R_i \times P_i \\ &\equiv (a, b, -p_u a - p_v b). \end{aligned} \tag{2}$$

The $\equiv$ symbol indicates the parallelism of the vectors, which implies componentwise equality up to a common scale. For $U = (u_x, u_y, u_z)$ and $V = (v_x, v_y, v_z)$, if $U \equiv V$ then $(u_x, u_y, u_z) = (\lambda v_x, \lambda v_y, \lambda v_z)$. The NIGP of a line is called the vanishing gradient of a line in [14]. Note that $(-p_u a - p_v b)$ is equal to c since $(p_u, p_v, 1)$ is on the line L_i. In fact c is proportional to the minimum distance of the line L_i from the origin of the $u - v$ plane. Then A is equal to

$$A = (a, b, c). \tag{3}$$

The NIGP, A, of the line L is a very simple quantity that we can obtain directly from the equation of L_i on the image plane.

3. CONSTRAINTS UNDER PERSPECTIVE PROJECTION

We now derive several constraints that follow from the properties of lines and surfaces under perspective projection.

3.1. Choosing a Representation for Surface Orientation

Consider a plane in 3-D having the equation

$$ax + by + cz + d = 0. \tag{4}$$

The normal of this plane is $N = (a, b, c)$. However, as the normal of the plane has only two degrees of freedom, we can normalize the normal vector as $N = (p, q, 1)$, where $p = a/c$ and $q = b/c$ (note that this excludes cases where $c = 0$). (p, q) can be viewed as a point in the *gradient space*. Gradient space has been useful for orthographic analysis since the degeneracies of gradient space (normals of the planes parallel to the z axis) also correspond to the degeneracies of the orthographic projection (those planes project as lines). However, planes that are unrepresentable by the gradient space may be present in perspective images. Unfortunately, there is no known representation for the normal of a plane having only two components such that the equation of the plane is linear in these components and it is able to represent planes of any orientation. We will derive our constraints first in

[1] If the line passes through the origin then it projects as a point on the image plane; therefore this plane is defined for every visible line on the image plane.

abstract vector notation and then give two different representations. The first representation is the regular gradient space; it has the advantage of simplicity but contains important singularities. The second one is just a regular vector (p, q, r) in 3-D with the constraint that

$$p^2 + q^2 + r^2 = 1. \tag{5}$$

This can represent the normal of any plane in 3-D, with the added complexity that Eq. (5), a quadratic equation, should be included among the equations to be solved.

3.2. Shared Boundary Constraint

This constraint relates the orientation of two planes intersecting using the image of the line of intersection. Similar results have been derived previously in [14]. Say two planes have normals N_1 and N_2; then the line, $L = Rt + P$, formed by intersection of these planes has orientation

$$R = N_1 \times N_2. \tag{6}$$

If the image $L_i = R_i t + P_i$ has the equation $au + bv + c = 0$ on the $u - v$ plane then the NIGP of the line is $A = (a, b, c)$. Also note that $A \equiv R \times P$, that is $A \perp R$; therefore

$$A \cdot R = 0$$
$$A \cdot N_1 \times N_2 = 0. \tag{7}$$

This is the shared boundary constraint in the form of a vector equation. Depending on the representation of N_1 and N_2 the final equation changes, but the vector equation remains the same. If the gradient space is used with $N_1 = (p_1, q_1, 1)$ and $N_2 = (p_2, q_2, 1)$ then the shared boundary constraint becomes

$$a(q_2 - q_1) - b(p_2 - p_1) + c(p_2q_1 - p_1q_2) = 0. \tag{8}$$

Here a, b, and c are known and unique quantities up to a scale factor. This equation defines a line in $p - q$ space when we fix one of the normals N_1 or N_2.

If the (p, q, r) representation is used, then $N = (p_1, q_1, r_1)$, $N_2 = (p_2, q_2, r_2)$, and the constraint equation is

$$a(q_1r_2 - q_2r_1) + b(r_1p_2 - p_1r_2) + c(p_1q_2 - q_1p_2) = 0 \tag{9}$$

This equation defines a plane in terms of (p_1, q_1, r_1) when (p_2, q_2, r_2) is fixed or vice versa. In this representation there are actually two more constraint equations which are obtained by substituting (p_1, q_1, r_1) and (p_2, q_2, r_2) in Eq. 5.

3.3 Parallel Lines

In orthographic projection, parallel lines in 3-D project into parallel lines in the image. This, in general, is not the case under perspective projection. However, *if* we are given the information that two image lines are in fact parallel in 3-D, we can infer some important information about their orientations.

THEOREM 1. *If two lines $L_1 = R_1t + P_1$ and $L_2 = R_2t + P_2$ are known to be parallel in 3-D (i.e., $R_1 \equiv R_2 \equiv R$), then the orientation of the lines, R, is given by $R = A_1 \times A_2$ where A_1 and A_2 are the NIGPs of the lines L_1 and L_2.*

Proof. This is not an entirely new result but has been used previously in [14, 15]. A_1 and A_2 are computable from the image of the lines L_1 and L_2 as given by Eq. (3). Also from Eq. (2), $A_1 = R \times P_1$ and $A_2 = R \times P_2$, then we get:

$$\begin{aligned} A_1 \times A_2 &= (R \times P_1) \times (R \times P_2) \\ &= (R \cdot (P_1 \times P_2))R - (P_1 \cdot (R \times R))P_2 \\ &= (R \cdot (P_1 \times P_2))R \\ &\equiv R. \end{aligned} \tag{10}$$

Unless the lines L_1 and L_2 are parallel to the image plane, their images intersect and the intersection point, I, is given by $I \equiv A_1 \times A_2$. Note that this theorem does not have any analogy in orthographic projection.

3.4. Orthogonality Constraint

This constraint is derived from the *knowledge* that two lines in a plane are orthogonal in 3-D. In orthographic projection, this hint may come from the observation ot a skew symmetry. For perspective projection, we will assume the orthogonality knowledge to be given for now. In the next subsection, we show how it may be inferred from a new form of symmetry that we call the *convergent symmetry*. This constraint could also be applied to curved surfaces where we may have some means of detecting lines of minimum and maximum curvatures.

Consider a plane Π having normal N, and two orthogonal lines on the plane

$$L_1 = R_1t + P_1 \quad L_2 = R_2t + P_2. \tag{11}$$

These lines have the NIGPs $A_1 = (a_1, b_1, c_1)$ and $A_2 = (a_2, b_2, c_2)$. Since A_1 is the normal of the plane containing L_1 and the origin and also L_1 is on the plane Π, L_1 is the intersection of these two planes. Therefore

$$R_1 = A_1 \times N \quad R_2 = A_2 \times N. \tag{12}$$

By the orthogonality constraint (i.e., $R_1 \perp R_2$) we get

$$R_1 \cdot R_2 = 0$$
$$(A_1 \times N) \cdot (A_2 \times N) = 0. \tag{13}$$

This is the orthogonality constraint in the form of a vector equation. This constraint takes slightly different forms depending on the representation used. If we use the gradient space, then $N = (p, q, 1)$ and the constraint equation is

$$(a_1a_2 + c_1c_2)q^2 + (b_1b_2 + c_1c_2)p^2 - (a_1b_2 + a_2b_1)pq \\ -(b_1c_2 + b_2c_1)q - (a_1c_2 - a_2c_1)p + b_1b_2 + a_1a_2 = 0. \tag{14}$$

This is a quadratic equation in terms of the gradient (p, q) of the plane Π. For most choices of parameters, this will represent a hyperbola on the $p - q$ plane, as for orthographic projection, but not necessarily centered at the origin.

If we use the (p, q, r) representation then $N = (p, q, r)$ and the constraint equation is

$$(b_1b_2 + a_1a_2)r^2 + (c_1c_2 + a_1a_2)q^2 + (c_1c_2 + b_1b_2)p^2 \\ -((b_1c_2 + b_2c_1)q - (a_1c_2 + a_2c_1)p)r \\ - (-a_1b_2 + a_2b_1)pq = 0. \tag{15}$$

Note that p and q in this representation are not the same as for the gradient space. This is a quadratic surface in $p - q - r$ space. As in the case of shared boundary constraint, this equation should be used in conjunction with the constraint Eq. (5) which is a sphere. With these constraints only one degree of freedom is left for the orientation of the plane Π.

As in the case of orthographic projection, the orthogonality constraint by itself does not give unique orientations, and some *ad hoc* choices could be made such as choosing the tips of constraint hyperbola [4].

3.5 Convergent Symmetry

In this section an object refers to a planar surface in 3-D bounded with a piecewise linear boundary, and a figure refers to the projection of the boundary. An object is called *symmetric* in 3-D if there are lines on the object joining the points of the boundary, called lines of symmetry, such that the locus of the midpoints of these lines forms another line, called the axis of symmetry, and the axis of symmetry is orthogonal to the lines of symmetry. An arrowlike object and its symmetry axis with the lines of symmetry are shown in Fig. 1(a). If we project a symmetric object using orthographic projection we get a figure having skew symmetry as proposed by Kanada [4]. If we use perspective projection then we get a figure having a new symmetry called *convergent symmetry*.

DEFINITION. A figure is said to be *convergent symmetric* if there exist point-to-point correspondences between all points of the figure such that

(a) All lines joining points of correspondence, called lines of symmetry, intersect in a common point on the image plane.

(b) The projections of the mid-points of the 3-d lines of symmetry lie along a straight line on the image plane.

Under perspective projection, projections of parallel lines meet at a point on the image plane. Therefore the projections of the lines of symmetry should meet at a point when extended on the image plane; that is, they should be convergent. The axis of symmetry is, however, no longer defined by the locus of the mid points of the lines of symmetry in the image plane. Instead, we require

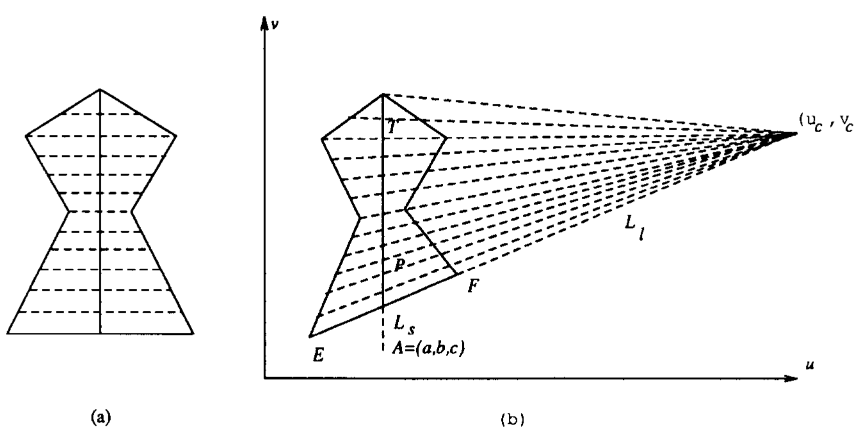

FIG. 1. (a) An arrowlike planar object with its axis of symmetry, solid vertical line, and lines of symmetry, dashed horizontal lines. (b) Projection of the arrowlike object and its convergent symmetry lines. Dashed lines are the lines of symmetry meeting at the point (u_c, v_c); L_l is one of the lines of symmetry meeting the boundary at points E and F. The vertical solid line is the axis of symmetry, L_s, having NIGP of $A = (a, b, c)$.

that the midpoints of the lines of symmetry, in 3-D be along a straight line. We show how this 3-D computation can be performed in the following. Figure 1 (b) shows an example of an arrowlike object under perspective projection with its axis and lines of symmetry.

The corresponding points would be easier to determine in a figure with several corners, as each corner must correspond to another corner. However, the above definition is general and applies to any figure (including curved figures). In general, of course, we can first choose the point of convergence, and then define lines of symmetry from it. The following procedure is to check whether the second part of the definition is also satisfied.

For every line of symmetry we can find the projection of its 3-D mid point. Consider Fig. 1 (b). Let $L_l = R_l t + P_l$ be one of the lines of symmetry, and let E and F be the two corresponding points on this line with image coordinates of (u_e, v_e) and (u_f, v_f), respectively. Let (u_c, v_c) be the point of convergence for the lines of symmetry. Then $R_l = (u_c, v_c, 1)$ from Theorem 1, and $P_l = (u_e, v_e, 1)$ as L_l passes through E. With these values for R_l and P_l the $u - v$ coordinates of the image of a point on the line L_l are given by

$$\left(\frac{u_e + tu_c}{1 + t}, \frac{v_e + tv_c}{1 + t}\right). \tag{16}$$

The t value that gives the point F on line L_l is given by the solution to the equation

$$h(u_f, v_f, 1) = R_l t + P_l, \tag{17}$$

where h is a constant, such that the above equation is satisfied only for a particular value of t and h. The above equation gives

$$t_{int} = \frac{hu_f - u_e}{u_c} = \frac{hv_f - v_e}{v_c} = h - 1 \tag{18}$$

Eliminating the constant h gives two solutions for t_{int}:

$$t_{int} = \frac{u_e - u_f}{u_f - u_c} \quad t_{int} = \frac{v_e - v_f}{v_f - v_c} \tag{19}$$

These two values for t_{int} are in fact the same since the point (u_f, v_f) is on the line defined by the ponts (u_e, v_e) and (u_c, v_c). The image coordinates for the projection of the midpoint of the line L_l between the points E and F is obtained by substituting $t = t_{int}/2$ into Eq. (16):

$$\left(\frac{(2u_e - u_c)u_f - u_c u_e}{u_f + u_e - 2u_c}, \frac{(2v_e - v_c)v_f - v_c v_e}{v_f + v_e - 2v_c}\right). \tag{20}$$

This gives us a procedure for finding the projection of the 3-D mid-point of any given line of symmetry. To check whether a given figure is convergent symmetric, we simply need to find the projections of mid-points of all lines of symmetry and check that they lie on a straight line, say L_s (L_s is the projection of the 3-D axis of symmetry). The NIGP value for L_s, $A = (a, b, c)$, can be obtained from

$$A = T \times P, \tag{21}$$

where T and P are midpoints of any two distinct lines of symmetry (the mid-points are given by Eq. 20).

3.5.1. *Computing Orientation Using Convergent Symmetry*

Now we will apply the constraint that the axis of symmetry is orthogonal to the lines of symmetry in 3-D. First, we state a theorem related to this.

THEOREM 2. *If a convergent symmetric figure is assumed to be a perspective projection of an orthogonal symmetric planar object, then the orientation of the planar object can be determined uniquely (unless the convergent symmetry is actually a skew symmetry with point of convergence at infinity, the axis of symmetry through the origin of the image plane and the lines of symmetry are orthogonal to the axis of symmetry on the image plane).*

We will give a constructive proof of this theorem in the following. Note that the theorem asserts that the constraints provided by convergent symmetry are much stronger than those provided by skew symmetry. The process is similar to that of skew symmetry analysis, but unlike orthographic projection, in perspective projection the axis of symmetry intersects every line of symmetry at a different angle than the others on the image plane. This results in a different constraint equation at every point on the axis of symmetry. Every equation gives a different constraint hyperbola on the $p - q$ plane (if $p - q$ space is used). But all of these hyperbolas pass through one point on the $p - q$ plane and this point is the only solution to all of these constraint equations. Therefore, we get a unique answer for the surface normal by using convergent symmetry, except for some special cases noted in the theorem. In gradient space (p, q) representation, we can find a closed form solution. And the degeneracies of $p - q$ space can be compensated as will be clear later. Consider the object in Fig. 1. The axis of symmetry has the NIGP of $A = (a, b, c)$ and assume that A is normalized (i.e., $|A| = 1$). There are infinitely many lines of symmetry all of which pass through the point (u_c, v_c) on the image plane. Say the intersection of these lines with the v axis has the coordinate $(0, k)$, where k is a parameter having a range that covers the figure. Then these lines have the NIGP

$$A_l = (0, k, 1) \times (u_c, v_c, 1) = (k - v_c, u_c, -u_c k). \quad (22)$$

Say the normal of the plane containing the object is $N = (p, q, 1)$; then from the orthogonality constraint we have

$$(A \times N) \cdot (A_l \times N) = 0, \quad (23)$$

$$k((-cq^2 + bq - cp^2 + ap)u_c + aq^2 - bpq - cp + a) + (-aq^2 + bpq + cp - a)v_c + ((-ap - c)q + bp^2 + b)u_c = 0. \quad (24)$$

This constrain should be satisfied independent of the value of k; then we get two constraints of the form

$$(-cq^2 + bq - cp^2 + ap)u_c + aq^2 - bpq - cp + a = 0$$
$$(-aq^2 + bpq + cp - a)v_c + ((-ap - c)q + bp^2 + b)u_c = 0. \quad (25)$$

There is only one real solution to these equations, given by

$$p = \frac{(ab^2 + a^3)v_c^2 + (-b^3 - a^2b)u_c v_c + (-b^2 - a^2)cu_c - ac^2 + a}{(b^2 + a^2)cv_c^2 + (bc^2 - b)v_c + (b^2 + a^2)cu_c^2 + (ac^2 - a)u_c}$$

$$q = -\frac{((ab^2 + a^3)u_c + (b^2 + a^2)c)v_c + (-b^3 - a^2b)u_c^2 + bc^2 - b}{(b^2 + a^2)cv_c^2 + (bc^2 - b)v_c + (b^2 + a^2)cu_c^2 + (ac^2 - a)u_c}. \quad (26)$$

This gives us the normal, $N = (p, q, 1)$, of the plane containing the object in terms of the observable, $A = (a, b, c)$, and the intersection point, (u_c, v_c), of lines of symmetry on the image plane. As mentioned before, in the gradient space representation the normals of the planes that are parallel to the z axis are not representable, because those planes have the third component of the normal vectors equal to zero, and the equivalent of a vector, $V = (f, g, l)$, under this representation is obtained by dividing the vector by the third component of the vector, $(f/l, g/l, 1)$. However, the expressions for p and q in Eq. (26) have the property that the denominator for p and q are the same then by multiplying the N vector with this denominator we get another vector, N', having the same orientation as N but have no singularity as for representing planes parallel to z axis. Then the vector N' is

$$N' = ((ab^2 + a^3)v_c^2 + (-b^3 - a^2b)u_c v_c + (-b^2 - a^2)cu_c - ac^2 + a, -((ab^2 + a^3)u_c + (b^2 + a^2)c)v_c + (-b^3 - a^2b)u_c^2 + bc^2 - b, (b^2 + a^2)cv_c^2 + (bc^2 - b)v_c + (b^2 + a^2)cu_c^2 + (ac^2 - a)u_c). \quad (27)$$

Unlike skew symmetry under orthographic projection, for a convergent symmetric figure in perspective projection we can compute the orientation of the planar surface uniquely. That is, we even do not have the Neckers reversal; this is also in agreement with human perception. For example, the cube in Fig. 2 can be reversed if one tries but the reversed figure does not look symmetric at all. Therefore if we want to bias towards symmetric objects then there is only one answer for a convergent symmetric figure. This is another instance in which the perspective projection can be used to give more information than the orthographic projection.

All of the above derivations assume that the intersection pont (u_c, v_c) is not at infinity. In the latter case, we can obtain the solution using the limits of the solutions. Let us say the slope of the lines of symmetry is m; then we can obtain the solution by replacing v_c by mu_c in Eq. (26) and taking the limit as u_c goes to infinity. Then the solution in 3-component vector form is

$$N' = (am^2 - bm, -am + b, cm^2 + c). \quad (28)$$

In the theorem of convergent symmetry, we mentioned that we get a unique orientation from convergent symmetry except in some special cases. In fact, if lines of symmetry are parallel to each other on the image plane, and image of the axis of symmetry is passing through the origin of the image plane (i.e. $c = 0$), and on the image plane the axis of symmetry is orthogonal to the lines of symmetry (i.e. $b/a = m$), then N' becomes a zero vector. However this requires a very specific viewing angle and thus can be ignored under normal viewing conditions. This is the case that convergent symmetry acts like the skew symmetry of orthographic projection, that is, now it is a constraint leaving one degree of freedom, which is

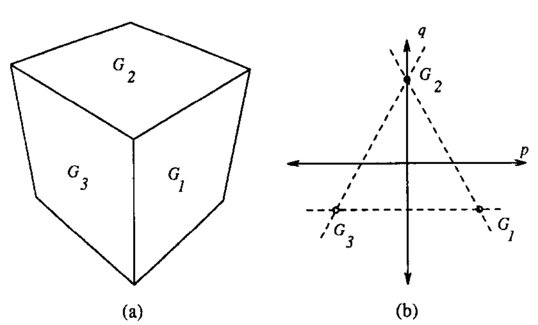

FIG. 2. A cube under perspective projection (a), and computed orientations for the faces shown as points on the $p - q$ space with the shared boundary constraints overlied dashed lines (b).

basically the orthogonality constraint given the equation 13.

4. USAGE OF THE CONSTRAINTS FOR POLYHEDRAL OBJECTS

In the previous section we have derived four constraints under perspective projection (however, not all four are independent as parallelism and orthogonality constraint are used in the convergent symmetry). For a given figure, we need to determine which constraints are applicable. Note that the shared boundary constraints make no regularity assumptions about the figure and must always apply (ignoring any "errors" in the line drawing).

Other constraints, however, require some regularity in the image and assume that the 3-D object obeys the corresponding regularity also. Of these regularities, symmetric convergence is quite stringent; i.e., it is unlikely to be caused by accident, though we still cannot guarantee that the 3-D object is orthogonally symmetric. Unfortunately, symmetric convergence is strong only when at least three lines converge on the image plane, as two lines always converge (unless they are parallel to each other on the image plane). Thus, a planar object having four sides to a face can always be construed to be symmetric convergent. Observations about parallelism and orthogonality may also not be apparent in the image. In orthographic projection, parallel lines remain parallel; in perspective projection they do not. On the other hand, however, in perspective projection we have much tighter constraints. Thus, one way to solve the interpretation problem is to *make* regularity assumptions and *verify* by using the constraints. We illustrate this by an example.

Figure 2(a) shows the image of a cube under perspective projection (the reader will get a better perception of the figure if the picture is held very close to the eye). Applying the shared boundary constraint (in the gradient space for the sake of illustration here) gives us a triangle, say $G_1G_2G_3$ in Fig. 2(b), which specifies the orientations of the three faces. Note that in perspective projection, the shape of the triangle may depend on both its position and its size (both of which need to be determined). Additional constraints can come from the symmetry of the faces. As described earlier, any quadrilateral can be viewed as being convergent symmetric. Assuming that the three faces are projections of orthogonally symmetric shapes (i.e., rectangles), we can get unique orientations for the three faces. In this example, these values happen to be consistent with the shared boundary constraints (and with the known correct answers from which the example was constructed). Alternately, we could have used the parallelism constraints between opposite sides of the faces. For this example, this regularity is suggested by the observation that groups of *three* lines (corresponding to parallel lines on two faces) do intersect in common points. Using this constraint, the answers turn out to be the same as before and hence consistent.

Of course, in general, we cannot expect all constraints to be satisfied simultaneously for all figures. If the image had been derived from a nonorthogonal prism, instead of a cube, the convergent symmetric results would not agree with the shared boundary constraints. Now, we can make several choices. We can either make all faces equally nonsymmetric (by some measure), or still achieve consistency by making two faces (any two) symmetric and the third nonsymmetric. In general, it should be possible to define a penalty function and find "optimal" solutions. However, we have not investigated such approaches. Our feeling is that the only time we can get strong interpretations is when some of the evidence is overwhelmingly strong and that this is the evidence we would use at the exclusion of the other constraints (those that require some assumptions).

5. EXTENSIONS TO CURVED SURFACES

In another paper we [12] have described methods for recovery of surface orientation of curved surfaces from contours under orthographic projection. The analysis was based on observation of a form of symmetry that we called *parallel symmetry*. Two planar curves are defined to be parallel symmetric if there exists a one-to-one correspondence between the points on the curves such that the tangents to the curves at corresponding points are parallel. The importance of the parallel symmetry is that, if we cut a zero Gaussian curvature surface with two parallel planes. Then it can be shown that we get two parallel symmetric curves from the intersection of the planes with the surface such that corresponding points of these curves are joined by the rulings of the zero Gaussian curvature surface.

Two curves that are parallel symmetric in 3-D also project into parallel symmetric curves in the image under orthographic projection. However, this is not the case under perspective projection. In the following, we give conditions that curves in a perspective image must satisfy if they are projections of parallel symmetric 3-D curves. Then we give a method for computing parallel symmetry for a certain class of images (those that are projections of conic surfaces) and show how this can be used for surface reconstruction.

5.1. PARALLEL SYMMETRY

Say there are two curves α_1 and α_2 in the image plane generated by the two planar 3-D curves β_1 and β_2 in planes parallel to a plane, call it Π, which passes through the origin. We have the relation

$$\alpha_1(s) = \frac{\beta_1(s)}{\beta_{1z}(s)}$$

$$\alpha_2(s) = \frac{\beta_2(s)}{\beta_{2z}(s)}, \quad (29)$$

where β_{iz} is the third coordinate of the curve β_i. The curves β_1 and β_2 are parallel symmetric if and only if we can form a monotonic correspondence function $f(s)$ such that

$$\beta_1'(s) = \beta_2'(f(s)), \quad (30)$$

where $\beta_i'(s)$ is the tangent vector of the curve $\beta_i(s)$. For each point of the curve α_i the NIGP of the tangent line (i.e., the line passing from the pont $\alpha_i(s)$ in the direction $\alpha_i'(s)$) of the curve is $A_i = \alpha_i(s) \times \alpha_i'(s)$. Since the curves β_1 and β_2 have parallel tangents at the corresponding points, from Theorem 1 the vector function $I(s) = A_1(s) \times A_2(f(s))$ gives the orientation of the tangents of the curves β_1 and β_2. That is,

$$\begin{aligned}\beta_1'(s) = \beta_2'(f(s)) \equiv I(s) &= A_1(s) \times A_2(f(s)) \\ &= (\alpha_1(s) \times \alpha_1'(s)) \times (\alpha_2(f(s)) \times \alpha_2'(f(s))). \quad (31)\end{aligned}$$

Since the curves β_1 and β_2 are planar curves resting on planes parallel to Π, their tangent vectors β_1' and β_2' must be parallel to Π, and the tangent vectors are on the plane Π. Therefore every orientation given by the function $I(s)$ (i.e., the vector from the origin to the points of $I(s)$) should be on the plane Π. The image of $I(s)$, $I_i(s)$, is the curve on the image plane that can be obtained by projecting the points of $I(s)$ on to the image plane. Since $I(s)$ is on the plane Π which passes through the origin its image has to be a line. Therefore, $I_i(s)$ being a line is the necessary condition for two curves α_1 and α_2 to be projections of the parallel symmetric curves β_1 and β_2. Also, the normal of the plane Π is just the NIGP of the line $I_i(s)$ since Π is the IGP of this line. Also, the line $I_i(s)$ is the locus of intersection points of the tangent lines of the curves $\alpha_1(s)$ and $\alpha_2(f(s))$. See Fig. 3.

5.2. Analysis of a Conic Surface

We now concentrate on conic surfaces (or linear straight homogeneous generalized cones in generalized cones terminology) cut by two parallel planes (say parallel to plane Π) to form curves, say $\beta_1(s)$ and $\beta_2(s)$. Let α_1 and α_2 be the projections of β_1 and β_2. In this case, the curves $\beta_1(s)$ and $\beta_2(s)$ are parallel symmetric; let the correspondence function be $f(s)$ as before. The lines joining the corresponding points on the curves β_1 and β_2 are the *rulings* of the surface. For a conic surface, these rulings intersect in a single point in 3-D. For a cylindrical surface, these rulings are parallel to each other. In either case, the projections of the rulings intersect at a point, say P, in the image plane.

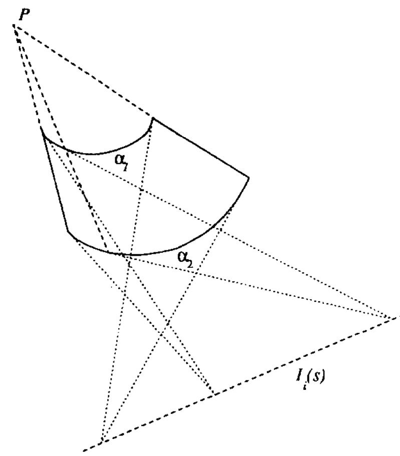

FIG. 3. Contours of a conic surface under perspective projection, with the point of convergence for the rulings P, and the line $I_i(s)$.

Point P can be found by the intersection of the lines joining the endpoints of α_1 and α_2. Now draw lines from P such that they intersect curves α_1 and α_2; the intersection points are the corresponding points on the two curves. With this correspondence we can construct the curve $I(s)$ and check if $I_i(s)$ is straight as in figure 3. If it is, we can interpret the figure as a conic surface. (Note: point P can also be found by a search process if the endpoints of the curves are not reliable.)

Now we have the plane Π containing $I(s)$, we can reconstruct β_1 and β_2 by back projecting α_1 and α_2 onto planes parallel to Π (up to a scale). However, this is not sufficient to reconstruct the conic surface; the distance between the planes containing the two curves still remains as one degree of freedom.

This degree of freedom can be fixed if we interpret the surface as being cylindrical (under perspective, a conic surface can always be interpreted as being cylindrical). Given the orientation N of the plane Π containing 3-D curves $\beta_1(s)$ and $\beta_2(s)$, the 3-D tangent $\beta_i'(s)$ is given by

$$\beta_i'(s) = A_i(s) \times N, \quad (32)$$

where $A_i(s)$ is the NIGP of the line passing through $\alpha_i(s)$ in the direction $\alpha_i'(s)$ $A_i(s) = \alpha_i(s) \times \alpha_i'(s)$; therefore

$$\beta_i'(s) = (\alpha_i(s) \times \alpha_i'(s)) \times N. \quad (33)$$

Now we have the orientation of $\beta_i(s)$ at any point. The orientation of the surface is given by

$$\beta_i'(s) \times R(s), \quad (34)$$

where $R(s)$ is the 3-D orientation of the rulings. Since the surface is cylindrical, $R(s)$ is constant (i.e., $R(s) = R$) and is equal to the intersection point P of the rulings on the image plane (cf. Theorem 1). That is $R \equiv P$ and the orientation of the surface at any point is given by

$$((\alpha_1(s) \times \alpha_1'(s)) \times N) \times P. \tag{35}$$

We conjecture that if the resulting surface corresponds to a right cylindrical surface[2] humans will accept this interpretation as being the most preferred. If the figure is to be interpreted as a noncylindrical conic surface, further assumptions need to be made. One alternative is to assume that the surface belongs to a right, generalized cone. We have not studied the human preferences in such cases, and such experiments are in fact difficult to perform.

6. CONCLUSION

We have derived some constraints on the interpretations of line drawings under perspective projection. Some of the constraints are analogous to the constraints used in orthographic analysis. However, in perspective analysis, we typically need to use one more variable in representing orientations; this makes some of the equations more complex and nonlinear. The observation of the regularities may also be not as clear with perspective as it is with orthographic projection. However, when such regularities can be inferred from some other, perhaps external, context, our constraints can be used directly.

Our major observation, however, is that when regularity is discovered in perspective, it provides much stronger constraints than under orthographic projection. We demonstrated this for the case of convergent symmetric figures. For the case of curved surfaces, too, perspective projection provides considerable amount of information. If the surface is cylindrical (or can be interpreted as being cylindrical), than we can reconstruct the surface just from its contours. For conic surfaces one degree of freedom is left if we do not make additional assumptions. We hope that these observations will lead to increased use, and exploitation, of perspective projection rather than perspective's being regarded as a complicating agent that can be ignored under "normal" viewing conditions.

[2] A right cylindrical surface is the one where planes cutting the surface to generate the curves β_1 and β_2 are orthogonal to the orientation of the rulings. This translates to the condition that $P \equiv N$, where N is the normal of the plane Π and P is the point on the image plane at which rulings intersect.

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