meeting at a small angle create a large junction[4], proportional to the ‘agreement’ of their respective tangents at that junction point[5]. In a similar way it is possible to generate all conics (see [66]) using only two repetitive masks (see Figure 6.31). When more then two masks are present (as in our actual implementation) many free-form curves are created. In practice our algorithm was tested on synthetic data composed of conics, sinusoids and free-form curves and was able to reconstruct all of them.

It is striking that the same general shape of the Extension field can be derived by two totally separate procedures. The same is true for the 3 dimensional Diabolo fields.

Williams [83], has shown that a random walk in 2-D space, where not only a position, but also a trajectory, are kept at every point, will yield a preferred direction everywhere in space. The shape of this vector field is surprisingly similar to ours.

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[4] Or a high density of ink on paper.

[5] In other words, we use our perceptual mechanisms to explain the behavior of the proposed computational algorithm.

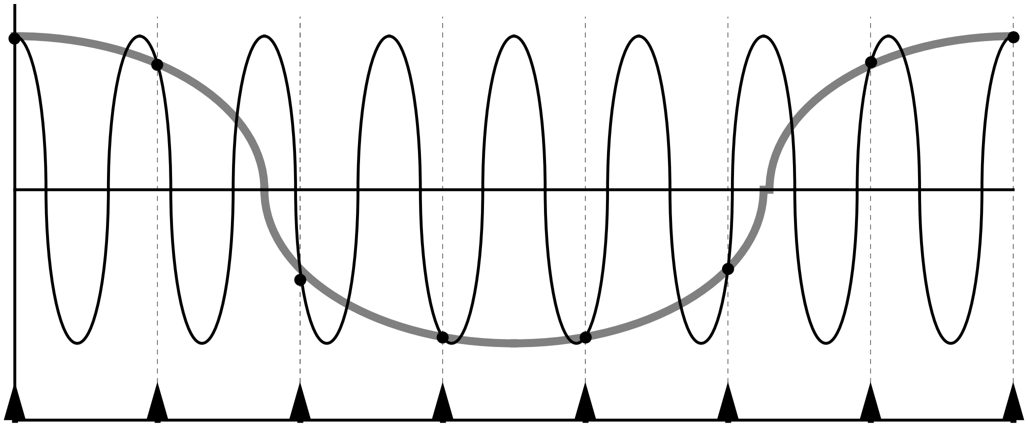

Figure 6.30 Moiré Patterns in 1-D. The original high frequency signal is sampled at low frequency creating a completely new signal (bold line).

In 2D, such an interference creates sets of co-curvilinear salient junctions (or windows) which we, as humans, tend to perceptually group into curves. Two lines

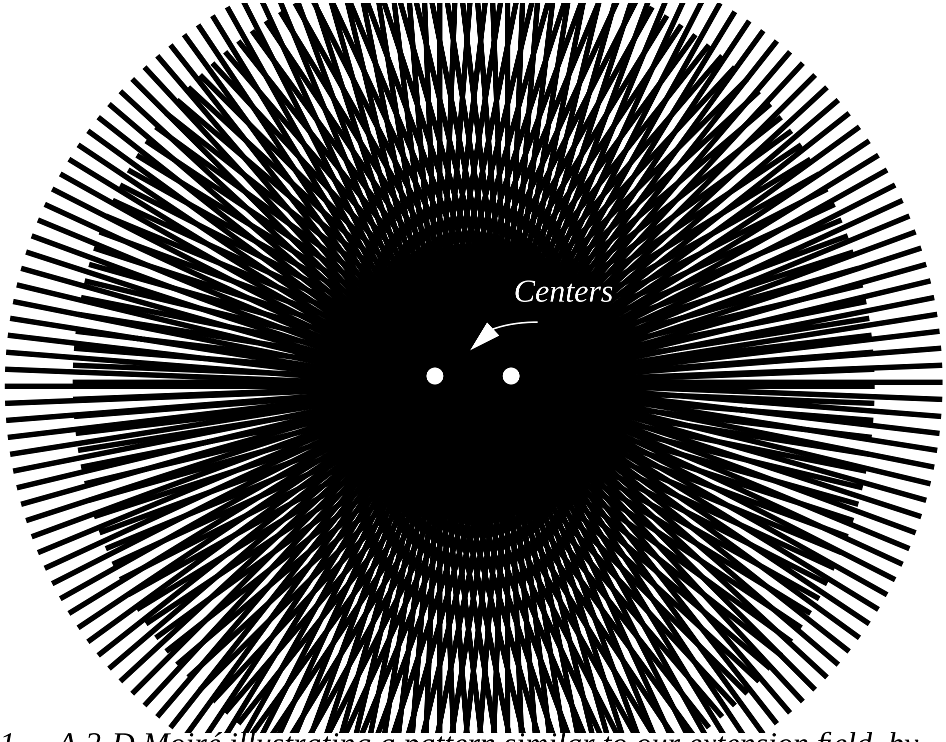

Figure 6.31 A 2-D Moiré illustrating a pattern similar to our extension field, by combining two radial repeating patterns situated side by side (point fields!). This is another way to construct the basic extension field.

For the general case, choose any point (say p) along the above circular arc. Obviously, the same circular arc minimizes the curve between points a and p. ❏

APPENDIX B

Other analogies

B.1 An interesting analogy to Moiré patterns

It may not be clear how arbitrary curves are created as a combination of semi-circular curves. Moiré patterns (e.g. [19]) and their formation behave in a similar way and give a good insight as to what is being computed in our system.

Moiré patterns are created when a repetitive signal is sampled at a sampling rate below the Nyquist rate[3], thus creating a new and apparently unrelated signal. A one dimensional example is given in Figure 6.30.

A way of generating *perceptual* Moiré patterns is by superimposing two (or more) repetitive patterns at a small angle. Such a combination is usually referred to as *interference* in physics.

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[3] This phenomena is usually called aliasing.

Proof:

According to Hölder's Inequality[2] [13], given that $1 \le p < \infty$, $1/p + 1/q = 1$, $\int_{\Upsilon} |x|^p \cdot dt < \infty$ and $\int_{\Upsilon} |y|^q \cdot dt < \infty$, then

$$\left(\int_{\Upsilon} |xy|\, dt \right) \le \left(\int_{\Upsilon} |x|^p dt \right)^{1/p} \left(\int_{\Upsilon} |y|^q dt \right)^{1/q} \tag{6.15}$$

Where equality holds only if p is 1, or if x=constant. and y=constant.

We make the following substitutions to equation (6.15),

$$\begin{aligned} x &= k(t) \\ y &= 1 \\ p &= \alpha \\ q &= \alpha/(\alpha - 1) \\ \Upsilon &= [0.\pi/2] \end{aligned} \tag{6.16}$$

after simplifications we get,

$$\int_0^{\pi/2} k(t)^{1+\alpha} dx \ge \frac{\pi}{2} \tag{6.17}$$

Again, equality holds only if either α=0 or *k(t)* is a constant. **This if true only if the curve is circular**. All other curves would have larger values of total curvature.

We have shown then, that a circular arc minimizes the total curvature for the scenario in Figure 6.29.

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[2] For p=q=2 this becomes the Schwarz (triangle) inequality.

for all values of α greater than 1. We first assume that the value of α is 1. It is obvious that all curves (assuming no change of curvature sign) will result in the same value for TC, namely the difference between start and end angles (which is π/2). However, we claim that a *circular arc* minimizes TC for any α *greater than* 1.

We choose a parameterization which keeps the boundaries of integration the same for all possible curves. The parameter chosen (*t*) is the accumulative angle along the curve. Also, without loss of generality, we consider a scenario where the interval of integration is 0..π/2, and a radius of 1 for the circular arc. By setting the start and end angles we ensure that the curve is tangent to the end points, as desired. Setting the radius to 1 makes the TC of the *circular* arc equal to π/2.

So, in general we need to show that,

$$\int_0^{\pi/2} k(t)^{(1+\alpha)} dt > \pi/2 \tag{6.13}$$

while

$$\int_0^{\pi/2} k(t)\, dt = \pi/2, \tag{6.14}$$

for α greater than 0.

APPENDIX A

Mathematical Treatment (Proofs)

A.1 Proof Of Lowest Curvature Connection (Special Cases Only)

Consider the simple scenario shown in Figure 6.29. We are looking for the lowest total curvature curve to connect the two segments. We claim that the circular arc connecting points *a* and *b*, minimizes the Total Curvature (TC)[1]

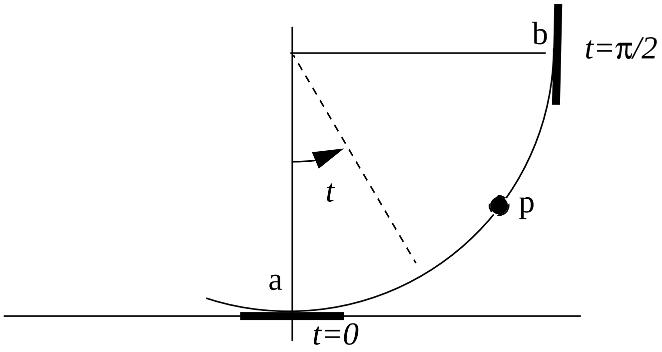

Figure 6.29 The parameter t grows from 0 to π/2 for all curves connecting and tangent to the two segments.

$$TC = \int_{\Upsilon} |k(t)|^{\alpha} dt \tag{6.12}$$

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[1] *k(t)* is the curvature function.

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