Surface matching with large deformations and arbitrary topology: a geodesic distance evolution scheme on a 3-manifold

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Abstract

The general problem of surface matching is taken up in this study. We propose a solution which handles the case of surfaces displaying large mutual deformations and arbitrary topological changes. The process described in this work hinges on a geodesic equation evolution for a family of surfaces embedded in the graph of a cost function. The cost function geometrically represents the matching criterion. This graph is a 3-manifold (or hypersurface) in 4-dimensionnal space, and the theory presented herein is a generalization of the geodesic curve evolution method introduced by R. Kimmel and al (reference). It also generalizes a 2D matching process developped by (reference Cohen-Herlin). An Eulerian formulation with level-sets of the geodesic surface evolution is also introduced, leading to a numerical scheme used for solving Hamilton-Jacobi (verifier avec Etienne si c'est vrai et citation Sethian) partial differential equations, which has proven to be very robust and stable. The theory we introduce in this paper clears the path for a generalization to higher dimensions, and brings a general theoretical setting for answering the very general problem of n-dimensionnal matching theory. The method is applied to both synthetic and real data, with examples showing both small and large deformations, and arbitrary topological changes.

Keywords: 3D matching, level-sets, geodesic evolution.

1 Introduction and previous work.

The problem of surface matching is a challenging question in both computer vision and graphics. In its most general setting, it consists in the following issue: given attributes found in different images, determine if the attributes come from a same primitive (ref Zhang). Applications are found in stereoscopic vision, robot navigation, segmentation etc. According to Zhang (ref Zhang), one can distinguish three main classes of matching problems:

- stereoscopic matching (see (reference 24 Zhang) for a general survey),
- object recognition (refs 8 18 Zhang),
- image sequence analysis.

Let us first focus on the matching problem in image sequence analysis. First approaches used iconic and pixel representations, and are called *template matching* techniques(ref 3 31 Zhang). Token based methods make use of simple geometric primitives (points, curves, surfaces) and take often the form of a graph matching problem, which is unfortunately known to be NP-complete (ref 77 Zhang). The general matching problem can be formulated in the terms of matching relationnal structures for which relaxation methods have been introduced (ref 42 Zhang). Tree based methods (ref 37 Zhang) tackle the problem using depth-first tree traversal algorithms. Matching 3D images has been studied for instance in (ref 81 Zhang) in the context of 3D stereoscopic reconstruction and this problem is important in 3D medical image analysis (ref 62 Zhang, trouver une ref d'Ayache). One can make use of special points coming from curvature information ((ref Sulger: I. Cohen, N. Ayache and P. Sulger: tracking points on deformable objects using curvature information. Proceedings of the Second European Conference on Computer Vision 1992, pp 458-466, Santa Margherita Ligure, Italy, May 1992,. ref 51 Zhang).

In computer graphics, the matching problem often takes the form of finding a morphing function between two objects, that is to say finding, for each point of a source object $\mathcal{S}$, a path which ends to another point on a destination object $\mathcal{D}$ (references siggraph). Here the matching criterion translates into geometric constraints on the path itself, which is a curve in the ambient space to which the two objects belong.

In this work, the general problem of matching two arbitrary surfaces $\mathcal{S}$ and $\mathcal{D}$ in $I\!R^3$ is contempled. A process for finding a function χ which associates, to any point of a source surface $\mathcal{S}$, a corresponding point on a destination surface $\mathcal{D}$ is presented. More precisely, we propose a method for computing paths between the two surfaces $\mathcal{S}$ and $\mathcal{D}$. Stated like that, the problem is quite ill-posed, and the function $\chi : \mathcal{S} \longrightarrow \mathcal{D}$ operates a matching according to specific criteria. The method uses a generalization of the geodesic curve propagation evolution introduced by R. Kimmel and al. in (ref Kimmel). It also generalizes the curve matching algorithm proposed in (ref Cohen-Herlin). The main idea of the method consists in:

1. building a cost hypersurface $W \subset I\!R^4$ between the two surfaces $\mathcal{S}$ and $\mathcal{D}$,
2. compute, using a level-set formulation, the evolution of a geodesic distance surface propagation scheme on the hypersurface W between $\mathcal{S}$ and $\mathcal{D}$.
3. Use the preceding geodesic evolution scheme to generate distance maps on W, and finally,
4. find the paths of minimal costs on W between $\mathcal{S}$ and $\mathcal{D}$.

The hypersurface W mentionned in step 1 incorporates the matching criterion between source and destination surfaces $\mathcal{S}$ and $\mathcal{D}$. This approach allow the design of matching criteria of different kinds: one may match the two surfaces according to distance, or to curvature. From this point of view, the method proposed in this study is a true generalization of the curve matching process introduced in (ref Cohen-Herlin). It needs, however, a generalization to 3-manifolds of a geodesic curve propagation scheme introduced by R. Kimmel and al. in order to complete step 2. This is a major contribution of this study, as such a generalization is far from trivial and necessitates the setting of a new theory developped in this presentation. Using exterior algebras and the Hodge * operator (ref Math) it also allows a generalization to higher dimensions. Hence we suggest that the matching formulation developped in this research brings a possible solution to a quite general class of surface matching problems according to geometric criteria, as the source and destination surfaces $\mathcal{S}$ and $\mathcal{D}$ may have different topologies and display large deformations.

To provide the reader an almost complete exposition of the method, we organize the presentation in the following manner. First, we set up the geodesic surface evolution method on a 3D hypersurface W embedded in $I\!R^4$. This theory makes use of the Hodge "star" (*) operator, a notion briefly reviewed in an appropriate subsection. Another subsection is devoted to the level-set formulation of that geodesic surface evolution schme, allowing for the computation of distance maps on the 3-manifold W. The level-set evolution equation takes the form of Hamilton-Jacobi partial differential equations, for which Sethian (ref bouquin Sethian) has introduced stable and robusts numerical resolution schemes described in another section. Then, the surface matching method is introduced, with the computation of paths between the source and destination surfaces $\mathcal{S}$ and $\mathcal{D}$ which minimize a cost function whose graph is W. A specific section is devoted to implementation matters and the study of synthetic and real examples. Lastly we contemplate a generalization of the method to higher dimensionnal manifolds. Then we conclude and sketch some perspectives.

2 A geodesic distance evolution rule for propagating surfaces on a 3-manifold.

In (ref Kimmel), it is proven that, if $C = \alpha(u, 0)$ is an initial curve curve parametrized by u, and lying on a surface in $I\!R^3$, the curve $\alpha(u, t)$ drawn on the same surface, and whose every point is at geodesic distance t from C is solution of the following partial differential equation:

$$\frac{\partial \alpha}{\partial t} = \vec{N} \otimes \vec{\tau^u} \tag{1}$$

where $\vec{\tau^u}$ is the tangent vector to α: $\vec{\tau^u} = \frac{\partial \alpha}{\partial u} / || \frac{\partial \alpha}{\partial u} ||$, $\vec{N}$ the normal to the surface, and the symbol $\otimes$ stands for the cross-product of two vectors in $I\!R^3$. The proof involves two preliminary results, the first one stating that the curve $\beta(t) = \alpha(u, t)|_{u=u_0 \text{ fixed}}$ is a geodesic for any value of u_0, and the second making use of the Gauss Lemma (ref Spivak) to complete the proof. As any attempt at solving equation 1 may turn out to be quite difficult in general (the curve is constrained to remain on a surface) the authors make use of a property coming from front propagation theory, allowing for the study of the normal speed of the projection $\pi \circ \alpha(u, t)$ where $\pi : I\!R^3 \longrightarrow I\!R^2$ is the standard projection on the (x, y) plane. For that matter, the surface on which the curves $\alpha(u, t)$ are lying is supposed to be a graph surface, i.e. the graph of a function $z(x, y)$, and it is shown that the normal speed V of the projected curve is given by the formula:

$$\begin{aligned} V &= \sqrt{\frac{n_1^2(1+q^2) + n_2^2(1+p^2) - 2pqn_1n_2}{(1+p^2+q^2)}} \\ &= \sqrt{an_1^2 + bn_2^2 - cn_1n_2} \end{aligned} \tag{2}$$

with $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$, and (n_1, n_2) are the coordinates of the normal vector to the projected curve. Then, using an eulerian approch, the projected curve $\pi \circ \alpha(u, t)$ is written as a level-set $\varphi^{-1}(0)$, and the evolution equation of function φ is shown to be:

$$\varphi_t = \sqrt{a\varphi_x^2 + b\varphi_y^2 - c\varphi_x\varphi_y}. \tag{3}$$

(with $\varphi_t = \frac{\partial \varphi}{\partial t}$ etc.) for which stable and powerful numerical resolution schemes have been proposed (ref Kimmel et Sethian).

Using these results (ref Cohen Herlin) build a curve matching process by setting up a cost surface incorporating geometric matching criteria (position in ambient space and curvature) and compute the paths of a matching function by integrating the differential equation:

$$\frac{\partial \gamma}{\partial s} = -\nabla(\varphi + \psi) \tag{4}$$

with φ and ψ being the two solutions of equation 3 starting from the source and destination curves respectively.

Our program is to generalize all this theory to the cases of surfaces, that is to say:

- find a geodesic distance evolution scheme, analogeous to 1, but for the case of surfaces lying on a 3-manifold embedded in 4-space.
- Set up a level-set formulation of the previous geodesic distance surface evolution scheme, which has numerous advantages, among them: allowing topological changes, giving up parametrization, easy building of distance maps, stable and robust numerical resolution schemes.
- Use the preceding formalism to build a surface matching process.

Let X be a 3-manifold (or hypersurface) in $I\!R^4$. We suppose X compact and path-connected.[1] From these assumptions, one can derive, using an easy application of the Hopf-Rinow-De Rham theorem (ref Spivak) that given any two points M_0 and M_1 in X, there is a unique path $\gamma : [0, 1] \longrightarrow X$ between M_0 and M_1 ($\gamma(0) = M_0$, $\gamma(1) = M_1$) whose length minimizes the lengths of all paths between the two points. The length of γ is called the geodesic distance between M_0 and M_1, and will be denoted $d_X(M_0, M_1)$ in the sequel. Moreover, the path γ is necessarily a geodesic curve on X, i.e. a curve such that the second derivative $\frac{d^2\gamma}{du^2}$ is always perpendicular to X. Let $\mathcal{Y} \subset \mathcal{X}$ be a surface (2-manifold) "drawn" on X. We consider the set of surfaces in X whose points are located at geodesic distance t from $\mathcal{Y}$. In other words, we consider the set:

$$\Xi_t = \{M \in X \mid d_X(M, \mathcal{Y}) = t\} \; . \tag{5}$$

For each value of t, Ξ_t is a surface on X, and we aim at finding a partial differential equation governing the evolution of Ξ_t as the parameter t evolves. For that matter, not only do we need a notion cross-product in 4-space, but also a means of deriving simple formulae about such a cross-product. These formulae are needed in the demonstration of intermediate propositions. The mathematical tool that can achieve these requirements is given by the Hodge * operator, a notion recalled the next subsection.

[1] These assumptions do not restrict the validity of the theory presented in this work, as the manifold X will appear as the graph of a cost function, which automatically satisfies these requierements in practice.

2.1 Exterior algebras and the Hodge * operator.

The theory is only briefly reviewed here. The reader is referred to (Marsden) for a complete exposition of the subject. Let E be a finite dimensional vector space over $I\!R$, and let $(e_1, e_2, \cdots, e_n)$ be a basis of E, with $n = \dim E$. For any integer p between 0 and n, one can construct new vector spaces, usually denoted $\Lambda^p(E)$, such that:

- by convention, $\Lambda^0(E) = I\!R$ and $\Lambda^1(E) = E$.
- $\Lambda^p(E)$ is the set of formal sums $\sum_{(i_1,i_2,\cdots i_p)} a_{i_1,i_2,\cdots i_p} u_{i_1} \wedge u_{i_2} \wedge \cdots u_{i_p}$, for mul-indexes $(i_1, i_2, \cdots i_p)$ and real coefficients $a_{i_1,i_2,\cdots i_p}$, the u_{i_j} being ordinary vectors of E.

The "wedge" products $u_{i_1} \wedge u_{i_2} \wedge \cdots u_{i_p}$ are supposed to be multilinear in the variables $u_{i_1}, u_{i_2}, \cdots, u_{i_p}$ and alternate, that is to say $u_{i_1} \wedge u_{i_2} \wedge \cdots \wedge u_{i_p} = 0$ if one of the vectors in this wedge product is equal to another vector. From these conditions, one can derive that for every p, $0 \leq p \leq n$, $\Lambda^p(E)$ is finite dimensional, with dimension $\begin{pmatrix} n \\ p \end{pmatrix} = \frac{n!}{p!(n-p)!}$. A basis of $\Lambda^p(E)$ is given by the family of vectors $(e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_p})$ with $1 \leq i_1 \leq i_2 \leq \cdots i_p \leq n$. For instance, if E is 4-dimensional space $I\!R^4$ with standard basis (e_1, e_2, e_3, e_4), the standard basis of $\Lambda^3(I\!R^4)$ is $(e_1 \wedge e_2 \wedge e_3, e_1 \wedge e_2 \wedge e_4, e_2 \wedge e_3 \wedge e_4, e_1 \wedge e_3 \wedge e_4)$, and the standard basis of $\Lambda^4(I\!R^4) \simeq I\!R^4$ is $(e_1 \wedge e_2 \wedge e_3 \wedge e_4)$.

Now suppose a dot product $< \bullet, \bullet >$ is defined in E. A general dot product, usually denoted $< \bullet, \bullet >_p$ can be defined in $\Lambda^p(E)$ by the formula:

$$< u_1 \wedge u_2 \wedge \cdots \wedge u_p, w_1 \wedge w_2 \cdots \wedge w_p >_p = \begin{vmatrix} < u_1, w_1 > & < u_1, w_2 > & \cdots & < u_1, w_p > \\ < u_2, w_1 > & < u_2, w_2 > & \cdots & < u_2, w_p > \\ \cdots & \cdots & \cdots & \cdots \\ < u_p, w_1 > & < u_p, w_2 > & \cdots & < u_p, w_p > \end{vmatrix} \quad (6)$$

In equation 6 the quantities $< u_i, w_j >$ inside the determinant are ordinary dot products in E. Since $\begin{pmatrix} n \\ p \end{pmatrix} = \begin{pmatrix} n \\ n-p \end{pmatrix}$, the two vector spaces $\Lambda^p(E)$ and $\Lambda^{(n-p)}(E)$ are isomorphic. The Hodge * operator provides a standard isomorphism between these two vector spaces. It is defined in the following manner. Let $\lambda \in \Lambda^p(E)$ and $\mu \in \Lambda^{(n-p)}(E)$ be two elements of $\Lambda^p(E)$ and $\Lambda^{(n-p)}(E)$ respectively. The image of λ by the Hodge operator, denoted $*\lambda$, belongs to $\Lambda^{(n-p)}(E)$: $*\lambda \in \Lambda^{(n-p)}(E)$ and is characterized by the equality:

$$\lambda \wedge \mu = < *\lambda, \mu >_{n-p} e_1 \wedge e_2 \wedge \cdots \wedge e_n \quad (7)$$

Let us now see how all of this operates in practice. Suppose for example that E is the standard 3-space $I\!R^3$, and that u and v are two vectors in $I\!R^3$. Using elementary calculus and equation **??**, one can easily show that $*(u \wedge v)$ is the standard cross-product in $I\!R^3$. If w is another third vector in $I\!R^3$, $*(u \wedge v \wedge w)$ is simply the determinant of the three vectors u, v, w. Now take $E = I\!R^4$ (it is the case that interests us in this work), and choose three vectors u, v and w in $I\!R^4$. It is easy to prove that $*(u \wedge v \wedge w)$:

- is a vector in $I\!R^4$ perpendicular to u, v and w,
- is such that the basis $(u, v, w, *(u \wedge v \wedge w))$ is positively oriented,
- has components $\left(-\begin{vmatrix} u_2 & u_3 & u_4 \\ v_2 & v_3 & v_4 \\ w_2 & w_3 & w_4 \end{vmatrix}, \begin{vmatrix} u_1 & u_3 & u_4 \\ v_1 & v_3 & v_4 \\ w_1 & w_3 & w_4 \end{vmatrix}, -\begin{vmatrix} u_1 & u_2 & u_4 \\ v_1 & v_2 & v_4 \\ w_1 & w_2 & w_4 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}\right)$ in the standard basis of $\Lambda^3(I\!R^4)$, with $u = u_1e_1 + u_2e_2 + u_3e_3 + u_4e_4$ and similarly for v,w.
- satisfies the equality: $\| *(u \wedge v \wedge w) \|^2 = \begin{vmatrix} <u,u> & <u,v> & <u,w> \\ <v,u> & <v,v> & <v,w> \\ <w,u> & <w,v> & <w,w> \end{vmatrix}$.

This last equality about the norm $\| *(u \wedge v \wedge w) \|^2$ will be very useful for us, and it generalizes the usual formula about the norm of the ordinary cross-product in 3-space.
With this notion of cross-product in 4-space given by the Hodge * operator, we can now derive the geodesic distance evolution scheme for the family of surfaces Ξ_t. This is presented in the following subsection.

2.2 The geodesic distance evolution equation.

We consider a local parametrization $\alpha(u, v, t)$ of the surface Ξ_t. We must, in order to easily find geodesics on X, restrict ourselves to local orthogonal parametrizations, i.e. parametrizations $\alpha(u, v, t)$ such that $<\vec{\tau^u}, \vec{\tau^v}>= 0$, with $\vec{\tau^u} = \frac{\alpha_u}{\|\alpha_u\|}$ and $\vec{\tau^v} = \frac{\alpha_v}{\|\alpha_v\|}$ (force instance, meridians and parallels of latitude on a map of the world form an orthogonal parametrization). It is an easy result that an orthogonal parametrization can always be found:

Lemma 1 *Around any point on a surface S, one can always find a local orthogonal parametrization.*

Proof. Let $\sigma(u, v)$ be an arbitrary local parametrization around a point in S. If $p \in S$ is in the image of σ, one can, in the tangent space $T_p(S)$ at p apply the Gram-Schmidt orthogonalization process to the "natural basis" of $T_p(S)$ given by $(\frac{\partial\sigma}{\partial u},\frac{\partial\sigma}{\partial v})$. Let us denote by $(\vec{\tau^u}(p),\vec{\tau^v}(p))$ the orthonormal basis of $T_p(S)$ obtained in this manner. The maps χ_1 and χ_2 defined by $\chi_1(p)=\vec{\tau^u}(p)$ and $\chi_2(p)=\vec{\tau^v}(p)$ are C^1, since the entre vectors never vanish and the Gram-Schmidt orthogonalization process is a C^1 function of the non-vanishing entry vectors. From the local existence and uniqueness theorem for ordinary differential equations, one can integrate the vector fields χ_1 and χ_2 to generate one-parameter groups of local diffeomorphisms ϕ_1 and ϕ_2. The local parametrization given by (ϕ_1,ϕ_2) is then orthogonal. □

So let $\alpha(u,v,t)$ be a local orthogonal parametrization of a family of surfaces in X, and let $\vec{\tau^u}=\frac{\alpha_u}{\|\alpha_u\|}$, $\vec{\tau^v}=\frac{\alpha_v}{\|\alpha_v\|}$ be the two unitary tangent vectors determined by α. Let $\vec{N}$ be the normal vector to X. We suppose that the family $\alpha(u,v,t)$ satisfies the following partial differential equation:

$$\frac{\partial\alpha}{\partial t}=*\left(\vec{N}\wedge\vec{\tau^u}\wedge\vec{\tau^v}\right)\tag{8}$$

We want to prove that, for each t, $\alpha(u,v,t)$ is a local parametrization of Ξ_t.

Let $\beta(t)$ be the curve in X defined by: $\beta(t)=\alpha(u,v,t)|_{u=u_0,v=v_0,\text{ fixed}}$. Then:

Lemma 2 *For any u_0,v_0, the curve $\beta(t)$ is a geodesic in X.*

Proof. We prove this lemma by showing that β_{tt} is perpendicular to $\vec{\tau^u}$, $\vec{\tau^v}$, and $*(\vec{N}\wedge\vec{\tau^u}\wedge\vec{\tau^v})$, $\vec{N}$ being the normal to X. The only possibility that will remain will be then: β_{tt} is colinear to $\vec{N}$ which means that β is a geodesic.

First note that

$$\beta_t=\frac{d\beta}{dt}=\frac{\partial\alpha}{\partial t}=*\left(\vec{N}\wedge\vec{\tau^u}\wedge\vec{\tau^v}\right)\ ,$$

so

$$\begin{aligned}
||\beta_t||^2 &= \left|\left|*\left(\vec{N}\wedge\vec{\tau^u}\wedge\vec{\tau^v}\right)\right|\right|^2\\
&= \begin{vmatrix}\langle\vec{N},\vec{N}\rangle & \langle\vec{N},\vec{\tau^u}\rangle & \langle\vec{N},\vec{\tau^v}\rangle\\ \langle\vec{\tau^u},\vec{N}\rangle & \langle\vec{\tau^u},\vec{\tau^u}\rangle & \langle\vec{\tau^u},\vec{\tau^v}\rangle\\ \langle\vec{\tau^v},\vec{N}\rangle & \langle\vec{\tau^v},\vec{\tau^u}\rangle & \langle\vec{\tau^v},\vec{\tau^v}\rangle\end{vmatrix}\\
&= \langle\vec{N},\vec{N}\rangle\begin{vmatrix}\langle\vec{\tau^u},\vec{\tau^u}\rangle & \langle\vec{\tau^u},\vec{\tau^v}\rangle\\ \langle\vec{\tau^v},\vec{\tau^u}\rangle & \langle\vec{\tau^v},\vec{\tau^v}\rangle\end{vmatrix}-\langle\vec{N},\vec{\tau^u}\rangle\begin{vmatrix}\langle\vec{\tau^u},\vec{N}\rangle & \langle\vec{\tau^u},\vec{\tau^v}\rangle\\ \langle\vec{\tau^v},\vec{N}\rangle & \langle\vec{\tau^v},\vec{\tau^v}\rangle\end{vmatrix}\\
&\quad+\langle\vec{N},\vec{\tau^v}\rangle\begin{vmatrix}\langle\vec{\tau^u},\vec{N}\rangle & \langle\vec{\tau^u},\vec{\tau^u}\rangle\\ \langle\vec{\tau^v},\vec{N}\rangle & \langle\vec{\tau^v},\vec{\tau^u}\rangle\end{vmatrix}\\
&= 1-\langle\vec{N},\vec{\tau^u}\rangle^2-\langle\vec{N},\vec{\tau^v}\rangle^2=1
\end{aligned}$$

since the parametrization is orthogonal. This proves:

$$\langle\beta_{tt},\beta_t\rangle=\left\langle\frac{d^2\beta}{dt^2},\frac{d\beta}{dt}\right\rangle=\left\langle\frac{d}{dt}\left(*\left(\vec{N}\wedge\vec{\tau^u}\wedge\vec{\tau^v}\right)\right),*\left(\vec{N}\wedge\vec{\tau^u}\wedge\vec{\tau^v}\right)\right\rangle=0.\tag{9}$$

Hence, β_{tt} is perpendicular to $*(\vec{N} \wedge \tau^{\vec{u}} \wedge \tau^{\vec{v}})$.

Now we note that

$$\left\langle \frac{d}{dt}\left(*\left(\vec{N} \wedge \tau^{\vec{u}} \wedge \tau^{\vec{v}}\right)\right), \tau^{\vec{u}} \right\rangle = \left\langle \beta_{tt}, \frac{\alpha_u}{||\alpha_u||} \right\rangle$$

but

$$\frac{d}{dt}\left\langle \alpha_t, \frac{\alpha_u}{||\alpha_u||} \right\rangle = \frac{d}{dt}\left\langle *\left(\vec{N} \wedge \tau^{\vec{u}} \wedge \tau^{\vec{v}}\right), \tau^{\vec{u}} \right\rangle = \frac{d}{dt}0 = 0.$$

From the equality

$$\frac{d}{dt}\left\langle \alpha_t, \frac{\alpha_u}{||\alpha_u||} \right\rangle = \left\langle \alpha_{tt}, \frac{\alpha_u}{||\alpha_u||} \right\rangle + \left\langle \alpha_t, \left(\frac{\alpha_u}{||\alpha_u||}\right)_t \right\rangle$$

we deduce

$$\left\langle \alpha_{tt}, \frac{\alpha_u}{||\alpha_u||} \right\rangle = -\left\langle \alpha_t, \left(\frac{\alpha_u}{||\alpha_u||}\right)_t \right\rangle$$

Letting $g = ||\alpha_u||$, one can write

$$\begin{aligned} \left(\frac{\alpha_u}{||\alpha_u||}\right)_t &= \frac{d}{dt}\left(\frac{\alpha_u}{||\alpha_u||}\right) \\ &= \frac{\alpha_{ut}g - \alpha_u g_t}{g^2} \\ &= \frac{\alpha_{ut}}{g} - \frac{\alpha_u g_t}{g^2} \end{aligned}$$

which gives

$$\begin{aligned} \left\langle \alpha_t, \left(\frac{\alpha_u}{||\alpha_u||}\right)_t \right\rangle &= \left\langle \alpha_t, \frac{\alpha_{ut}}{g} - \frac{\alpha_u g_t}{g^2} \right\rangle \\ &= \left\langle \alpha_t, \frac{\alpha_{ut}}{g} \right\rangle - \left\langle \alpha_t, \frac{\alpha_u g_t}{g^2} \right\rangle \\ &= \frac{1}{g}\langle \alpha_t, \alpha_{ut} \rangle - \frac{g_t}{g^2}\left\langle *\left(\vec{N} \wedge \tau^{\vec{u}} \wedge \tau^{\vec{v}}\right), \tau^{\vec{u}} \right\rangle \\ &= \frac{1}{g}\frac{1}{2}\frac{d}{du}\langle \alpha_t, \alpha_t \rangle - \frac{g_t}{g^2}0 \\ &= \frac{1}{g}\frac{1}{2}\frac{d}{du}1 \\ &= 0, \end{aligned}$$

We can conclude that

$$\left\langle \beta_{tt}, \tau^{\vec{u}} \right\rangle = 0, \tag{10}$$

and similarly

$$\left\langle \beta_{tt}, \tau^{\vec{v}} \right\rangle = 0. \tag{11}$$

The curve $\beta(t)$ is a geodesic in X. □

Lemma 3 *The evolution of the family of surfaces Ξ_t is given by the equation*

$$\frac{\partial \alpha}{\partial t} = *\left(\vec{N} \wedge \tau^{\vec{u}} \wedge \tau^{\vec{v}}\right) \tag{12}$$

The proof of this lemma is omitted here, as it is a simple adaptation, using the the general Gauss Lemma (ref Spivak), of the proof given in (ref Kimmel).

In the next subsection, we compute the normal speed of the projection of Ξ_t onto the (x, y, z) hyperplane in $I\!R^4$.

2.3 Normal evolution of the projection of Ξ_t onto the (x, y, z) hyperplane.

In an attempt to generalize the 2D case, we now make the assumption that X is a graph hypersurface, that is to say $X = \{(x, y, z, w(x, y, z)\}$ for a function $w : I\!R^3 \longrightarrow I\!R$. Let $\pi : I\!R^4 \longrightarrow I\!R^3$ be the canonical projection onto the (x, y, z) hyperplane in $I\!R^4$ and let $\mathcal{S}(t)$ be the projection of the image of $\alpha(u, v, t)$ (that is to say, Ξ_t) onto that hyperplane:

$$\mathcal{S}(t) = \pi \circ \alpha.$$

We denote by p, q, r the following quantities: $p = \frac{\partial w}{\partial x}$,$q = \frac{\partial w}{\partial y}$ and $r = \frac{\partial w}{\partial z}$. Starting from a result mentionned in (ref front propagation), we admit that the trace of a propagating surface may be determined only by its normal velocity, as the other components of the velocity influence only the local parametrization. Our goal is then to compute the projected velocity of the evolving equal surface distance contour $V = \langle \pi \circ \alpha_t, \vec{n} \rangle$, $\vec{n} = (n_1, n_2, n_3)$ being the normal to the projected surface $\pi \circ \alpha(u, v, t)$. One can prove the following

Lemma 4 *The projected surface $\mathcal{S}(t)$ satisfies the normal propagation rule:*

$$\frac{\partial \mathcal{S}}{\partial t} = V \vec{n} \tag{13}$$

with

$$\begin{aligned} V &= \sqrt{\frac{(1+q^2+r^2)n_1^2 + (1+p^2+r^2)n_2^2 + (1+p^2+q^2)n_3^2 - 2pqn_1n_2 - 2prn_1n_3 - 2prn_2n_3}{1+p^2+q^2+r^2}} \\ &= \sqrt{an_1^2 + bn_2^2 + cn_3^2 - dn_1n_2 - en_1n_3 - fn_2n_3} \end{aligned} \tag{14}$$

Sketch of the proof. We only give the main steps of the proof, as it is rather long and tedious. The reader interested in writing a complete proof is strongly suggested to use a formal mathematical computation software. We set $\mathcal{S}(t) = \{(x(u, v), y(u, v), z(u, v), w(u, v))$. Using the formulae referenced in the section devoted to the Hodge * operator, one finds that the (x, y, z) components of $*\left(\vec{N}\wedge\vec{\tau^u}\wedge\vec{\tau^v}\right)$ are

$$\left(\begin{array}{c}
\dfrac{pq\begin{vmatrix} x_v & x_u \\ z_v & z_u \end{vmatrix}+q^2\begin{vmatrix} y_v & y_u \\ z_v & z_u \end{vmatrix}-rp\begin{vmatrix} y_u & y_v \\ x_u & x_v \end{vmatrix}-r^2\begin{vmatrix} y_u & y_v \\ z_u & z_v \end{vmatrix}-\begin{vmatrix} y_u & y_v \\ z_u & z_v \end{vmatrix}}{\sqrt{1+p^2+q^2+r^2}\sqrt{x_u^2+y_u^2+z_u^2}\sqrt{x_v^2+y_v^2+z_v^2}} \\[2ex]
\dfrac{-p^2\begin{vmatrix} x_v & x_u \\ z_v & z_u \end{vmatrix}-pq\begin{vmatrix} y_v & y_u \\ z_v & z_u \end{vmatrix}+rq\begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}+r^2\begin{vmatrix} x_u & x_v \\ z_u & z_v \end{vmatrix}+\begin{vmatrix} x_u & z_u \\ x_v & z_v \end{vmatrix}}{\sqrt{1+p^2+q^2+r^2}\sqrt{x_u^2+y_u^2+z_u^2}\sqrt{x_v^2+y_v^2+z_v^2}} \\[2ex]
\dfrac{p^2\begin{vmatrix} y_u & y_v \\ x_u & x_v \end{vmatrix}+pr\begin{vmatrix} y_u & y_v \\ z_u & z_v \end{vmatrix}-q^2\begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}-qr\begin{vmatrix} x_u & x_v \\ z_u & z_v \end{vmatrix}-\begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix}}{\sqrt{1+p^2+q^2+r^2}\sqrt{x_u^2+y_u^2+z_u^2}\sqrt{x_v^2+y_v^2+z_v^2}}
\end{array}\right)$$

which gives

$$\pi\circ\left(*\left(\vec{N}\wedge\vec{\tau^u}\wedge\vec{\tau^v}\right)\right) = \frac{\begin{pmatrix} qz_uw_v-qw_uz_v-ry_uw_v+ry_vw_u+y_vz_u \\ -qz_uw_v+qw_uz_v+ry_uw_v+y_uz_v-ry_vw_u-y_vz_u \\ py_uz_v-py_vz_u-qx_uz_v+rx_uy_v+qx_vz_u-rx_vy_u \end{pmatrix}}{\sqrt{1+p^2+q^2+r^2}\sqrt{x_u^2+y_u^2+z_u^2}\sqrt{x_v^2+y_v^2+z_v^2}}$$

The computation of the scalar product leads to

$$V=\langle\Pi\circ\alpha_t,\vec{n}\rangle= \frac{n_1^2n(-1-r^2-q^2)+n_2^2n(-1-r^2-p^2)+n_3^2n(-1-p^2-q^2)+2n_1n_2npq+2n_1n_3nrp+2n_2n_3nrq}{\sqrt{1+p^2+q^2+r^2}\sqrt{x_u^2+y_u^2+z_u^2}\sqrt{x_v^2+y_v^2+z_v^2}}$$

the quantity n being the norm of $\vec{n}$. But using the fact that

$$\left\langle\vec{\tau^u},\vec{\tau^v}\right\rangle=x_ux_v+y_uy_v+z_uz_v+w_uw_v=0,$$

we set

$$\begin{aligned} A &= (x_u^2+y_u^2+x_u^2+z_u^2w_u^2)(x_v^2+y_v^2+x_v^2+z_v^2w_v^2) \\ &= (x_u^2+y_u^2+x_u^2+z_u^2w_u^2)(x_v^2+y_v^2+x_v^2+z_v^2w_v^2) \\ &\quad -(x_ux_v+y_uy_v+z_uz_v+w_uw_v)^2\,. \end{aligned}$$

and also

$$B = n_1^2n(1+r^2+q^2)+n_2^2n(1+r^2+p^2)+n_3^2n(1+p^2+q^2)$$

One finds

$$\begin{aligned} A-B &= 2pqn_1n_2n^2-2prn_1n_3n^2-2qrn_2n_3n^2 \\ &= \left(\sqrt{1+p^2+q^2+r^2}\sqrt{x_u^2+y_u^2+z_u^2}\sqrt{x_v^2+y_v^2+z_v^2}\right)^2-B, \end{aligned}$$

so that

$$
\begin{aligned}
V &= \sqrt{\frac{(1+q^2+r^2)n_1^2+(1+p^2+r^2)n_2^2+(1+p^2+q^2)n_3^2-2pqn_1n_2-2prn_1n_3-2prn_2n_3}{1+p^2+q^2+r^2}} \\
&= \sqrt{an_1^2+bn_2^2+cn_3^2-dn_1n_2-en_1n_3-fn_2n_3}
\end{aligned} \tag{15}
$$

which complete the proof. □

2.4 Level-set formulation.

2.5 Distance maps on a 3-manifold.

3 Numerical resolution.

4 A general surface matching process.

5 Examples and implementation.

6 Generalization to higher dimensions.

7 Conclusion.